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A topological quantum field theory construction on piecewise linear manifolds

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Abstract

The generalisation of the topological quantum field theory construction of Turaev and Viro to arbitrary dimension is presented, and it is shown that q-deformed spin-networks, or the recoupling theory of the quantum group $U_q sl(2)$ provide a realisation of the initial data for the construction of 2-, 3- and 4-dimensional TQFTs.

Keywords: Simplicial manifolds, q-deformed spin-networks, Topological quantum field theory
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1. Introduction

Topological quantum field theories (TQFTs) are in many ways the simplest quantum field theories. They are QFTs in which the correlators, which completely characterise the theory, depend only on the topology of the underlying space. They hence generate global invariants (smooth invariants) of manifolds. The general characteristics of TQFTs mean that it is possible to write down a short list of axioms which reflect the essential properties of the formal path integrals and put the subject on a firm mathematical footing. This was first done by Atiyah in Ref. [1]. Atiyah's main aim in formulating such axioms was to encourage topologists to extract and build on the vast resources of QFT knowledge. However, they also suggest other possible constructions of TQFTs. It is apparent that as we are only interested in the global topology of the manifolds we may just as well work in the category of piecewise linear manifolds and carry out a con-

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struction in a purely combinatoric fashion, resulting in combinatoric invariants of manifolds such as the well known combinatoric formula for the Euler characteristic of a triangulated manifold. The Euler characteristic of course classifies two-dimensional manifolds topologically; however, in higher dimensions it is a very weak (highly degenerate) invariant. In searching for new combinatoric invariants we have theorems at our disposal [2,3] which reduce topological invariance, here meaning homeomorphisms between the piecewise linear manifolds, to local moves on a cell decomposition. It was physicists who first noticed that the 6*j*-symbols of SU(2) had the symmetries of a tetrahedron and that there were identities between sums of them, such as the Biedenharn–Elliot relation, which correspond to topology conserving moves on simplicial 3-manifolds. In the 60's Ponzano and Regge [4] noticed, from their asymptotic large spin formula for these 6*j*-symbols, an intriguing relationship between 6*j*-symbols and three-dimensional quantum gravity. In the early 70's Penrose [5] also found that SU(2) recoupling theory (spin networks) had some intrinsic properties of a three-dimensional space. However, the expression for the partition function given in Refs. [4] and [6], of a product of 6*j*-symbols for each tetrahedron summed over representations, though formally a topological invariant, is divergent. It was not until quantum groups and their representation theory emerged that Turaev and Viro were able to rigorously define such combinatoric invariants of 3-manifolds and a three-dimensional TQFT using the quantum 6*j*-symbols of $U_q sl(2)$ at q a root of unity. Generalisations of the Turaev–Viro state-sum model describing other three-dimensional TQFTs have been looked at in Refs. [7–9].

In Ref. [10] Boulatov uncovered a formal relationship between these three-dimensional state-sum models at $q=1$, and lattice gauge theory. This suggested a generalisation to arbitrary dimension, as noted in Ref. [11], where the $d=4$ case is discussed. This paper is concerned with the rigorous construction of topological state-sum models for general dimension. We now provide a short overview.

Overview. In Section 2 we describe a construction of topological quantum field theories (TQFTs) as axiomatised by Atiyah in Ref. [1], in terms of certain abstract initial data.

The construction should be seen as a generalisation of the three-dimensional "state-sum" model construction of Turaev and Viro in Ref. [12]. The approach used resembles that of Ref. [8], where one starts with a non-topological theory defined on simplicial manifolds using finite initial data and analyse the constraints that topological invariance imposes on this data. Hence we start by defining a simplicial quantum field theory (SQFT) as consisting of a finite dimensional vector space $V(S)$ associated with each $(d-1)$-dimensional simplicial manifold S and a linear map $Z(M): V(\partial M) \to \mathbb{C}$ with each d-dimensional simplicial manifold M, together with some "gluing" axioms. The abstract data that is introduced consists basically of Z and V for a d-simplex and $(d-1)$-simplex, respectively. The gluing axioms can be used to build up Z and V for all simplicial manifolds. A SQFT is said to be topological (and hence a TQFT) when the maps Z depend only on the topology of M. We use a theorem due to Pachner [3] to express this topological invariance in terms of constraints on the initial data of the SQFT. In order to make these constraints more transparent we translate the construction to one on the lattice dual to the triangulation. In dimension $d \leqslant 4$, the constraints can then be understood as identities between topological invariants of certain graphs embedded in a $(d-1)$-dimensional sphere. A realisation of TQFT initial data hence corresponds to finding examples of graph invariants satisfying these identities.

The first non-trivial realisation of the initial data for TQFTs of dimension 2, 3 and 4 is obtained from q-deformed spin-networks, when q is a root of unity. These networks correspond to the recoupling theory of the quantum group $U_q\mathrm{sl}(2)$, though in order to avoid unnecessary quantum group representation theory in Section 3 we describe the initial data explicitly in terms of the Kauffman bracket polynomial in q and q^{-1}.

2. The construction

2.1. The piecewise linear category

As this whole construction takes place in the piecewise linear category, we will first quickly run over some relevant definitions.

2.1.1. Simplicial complexes and piecewise linear maps

Definition 2.1. Given $n+1$ points $x_0, ..., x_n$ in general position in $\mathbb{R}^N$ for $N>n$, an n-simplex $\sigma^n \equiv (x_0, ..., x_n)$ with vertices $x_0, ..., x_n$ is defined to be the following subspace of $\mathbb{R}^N$:

$$\sigma^n \equiv \sum_{i=0}^{n} \lambda_i x_i , \quad \text{for} \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geqslant 0 . \tag{2.1}$$

Definition 2.2. A face (respectively proper face) of an n-simplex σ^n is any simplex whose vertices are a subset (respectively proper subset) of those of σ^n.

Definition 2.3. A simplicial complex K is a finite collection of simplexes in $\mathbb{R}^N$ such that if $\sigma_1^n \in K$ then so are all of its faces, and if $\sigma_1^n, \sigma_2^m \in K$ then $\sigma_1^n \cap \sigma_2^m$ is either a face of σ_1^n or empty.

Definition 2.4. A simplicial map $f: K \to L$ between two simplicial complexes K and L is a continuous map $f: |K| \to |L|$ which takes n-simplexes to n-simplexes for all

n. By $|K|$ we mean the following subset of $\mathbb{R}^N$:

$$|K| \equiv \bigcup_{\sigma \in K} \sigma . \tag{2.2}$$

f is a simplicial isomorphism if $f^{-1}: L \to K$ is also a simplicial map.

Definition 2.5. A subdivision K' of K is a simplicial complex such that $|K'| = |K|$, and each n-simplex of K' is contained in an n-simplex of K.

Definition 2.6. A piecewise linear homeomorphism $f: K \to L$ between two simplicial complexes is a map which is a simplicial isomorphism for some subdivisions K' and L' of K and L.

The simplicial approximation theorem says that any continuous map between $|K|$ and $|L|$ is homotopic to a simplicial map between some subdivisions K' and L' of the two simplicial complexes K and L. Also if there exists a piecewise linear homeomorphism between K and L, then $|K|$ and $|L|$ are homeomorphic. The equivalence relation generated by piecewise linear homeomorphisms defines topological equivalence of piecewise linear complexes.

2.1.2. Piecewise linear manifolds

Definition 2.7. The star of a simplex σ_1 in a simplicial complex K, denoted star (σ_1), is the union of all simplexes $\sigma_2 \in K$ satisfying $\sigma_2 \cap \hat{\sigma}_1 \neq \emptyset$, where $\hat{\sigma}_1$ is σ_1 without its proper faces.

Definition 2.8. The link of a simplex σ_1 in K is the union of all faces σ_3 of all simplexes $\sigma_2 \in \text{star}(\sigma_1)$ satisfying $\sigma_3 \cap \sigma_1 = \emptyset$ (see Fig. 1).

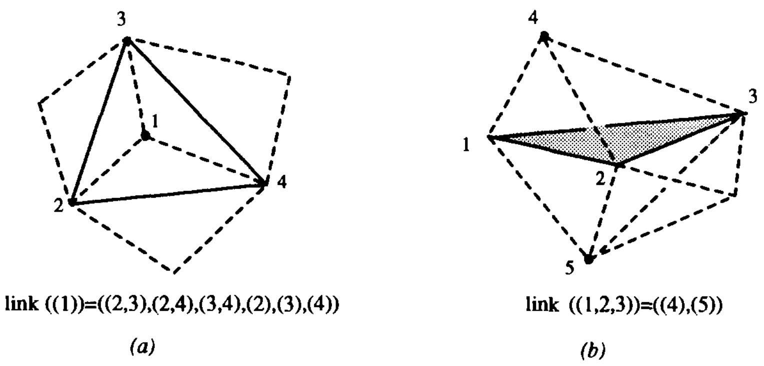


Fig. 1. (a) The link of a 0-simplex in a simplicial 2-manifold, and (b) the link of a 2-simplex in a simplicial 3-manifold.

Let Δ^n be a simplicial complex consisting of an n-simplex together with all its faces and let $\partial\Delta^n$ be Δ^n without the n-simplex. Then a simplicial complex K is called a simplicial n-manifold if for all r-simplexes $\sigma^r \in K$, $\operatorname{link}(\sigma^r)$ has the same topology as $\partial\Delta^{n-r}$ or Δ^{n-r-1}. The collection of simplexes of the first type form the set $\operatorname{Int} K$, and those of the second type form the complex ∂K. The equivalence classes of simplicial manifolds generated by piecewise linear homeomorphisms are called piecewise linear n-manifolds (or just n-manifolds) and for a given simplicial n-manifold M we denote by $\mathcal{M}$ the equivalence class to which it belongs.

2.2. Topological quantum field theory

A natural step in this topological quantum field theory construction is to first construct what we call a simplicial quantum field theory. This is defined so as to depend only on the connecting structure of simplicial manifolds, i.e. on simplicial manifolds up to simplicial isomorphisms.

2.2.1. Simplicial quantum field theory

A d-dimensional SQFT consists of a finite dimensional vector space $V(S)$ over $\mathbb{C}$ associated with each closed simplicial $(d-1)$-manifold S and linear maps $Z(M): V(\partial M) \to \mathbb{C}$ for each simplicial d-manifold. These should satisfy the following conditions:

(S1) V and Z depend on S and M only up to simplicial isomorphisms, i.e., $V(S_1)$ is isomorphic to $V(S_2)$ if S_1 and S_2 are related by a simplicial isomorphism, and similarly $Z(M_1) = Z(M_2)$, up to the isomorphism between $V(\partial M_1)$ and $V(\partial M_2)$, if M_1 and M_2 are related by a simplicial isomorphism.

(S2) $V(S_1 \cup S_2) = V(S_1) \otimes V(S_2)$, for the disjoint union of closed simplicial $(d-1)$-manifolds S_1 and S_2.

(S3) If S^* is S with its orientation flipped, then $V(S^*)$ is the dual space of $V(S)$, i.e., there is a non-degenerate natural pairing $\langle\ ,\ \rangle : V(S^*) \otimes V(S) \to \mathbb{C}$.

(S4) Z is multiplicative in that for the disjoint union of simplicial d-manifolds, M_1 and M_2

$$Z(M_1 \cup M_2)\cdot(v_1 \otimes v_2) = Z(M_1)\cdot v_1\, Z(M_2)\cdot v_2 ,$$

where $v_i \in V(\partial M_i)$. Also if $\partial M = S \cup \hat{S} \cup \hat{S}^*$ then (S, $\hat{S}$, $\hat{S}^*$ being disjoint)

$$Z(\hat{M})\cdot v = \sum_\alpha Z(M)\cdot(v \otimes e_\alpha \otimes e_\alpha^*) ,$$

where $\hat{M}$ is M with $\hat{S}$ and $\hat{S}^*$ identified in the natural way, $v \in V(S)$ and e_α and e_α^* are bases of $V(\hat{S})$ and $V(\hat{S}^*)$ satisfying $\langle e_\alpha^*, e_\beta\rangle = \delta_{\alpha,\beta}$.

A SQFT may be constructed with the following abstract data:

(D1) A finite set $I = \{a, b, c, ...\}$, called the colour set, containing a preferred element, 0, and an involution map $* : I \to I : a \to a^*$, with $0^* = 0$.

(D2) A non-zero gluing coefficient $\omega_a \in \mathbb{C}$, for each $a \in I$.

(D3) A finite dimensional vector space $V^{a_1 a_2 \cdots a_d}$ over the complex numbers $\mathbb{C}$, for each d-tuple $(a_1, ..., a_d) \in I^d$ with

$$V^{a_1 \cdots a_d} \equiv V^{b_1 \cdots b_d}, \quad \text{if } a_1 \cdots a_d \text{ is an even permutation of } b_1 \cdots b_d,$$

$$V^{a_1^* \cdots a_d^*} \equiv (V^{b_1 \cdots b_d})^*, \quad \text{if } a_1 \cdots a_d \text{ is an odd permutation of } b_1 \cdots b_d,$$

where $(V^{b_1 \cdots b_d})^*$ denotes the space dual to $V^{b_1 \cdots b_d}$, and we again denote the natural inner product by $\langle\ ,\ \rangle : V^{a_1 \cdots a_d} \otimes (V^{a_1 \cdots a_d})^* \to \mathbb{C}$.

(D4) A linear map $Z(\Delta^d) : V(\partial \Delta^d) \to \mathbb{C}$, for one particular fixed Δ^d (a d-simplex together with its faces).

We now describe the SQFT construction.

Definition 2.9. A k-colouring a of a simplicial n-manifold M is a map from the set of k-simplexes of M to the colour set I.

Definition 2.10. An ordered simplicial n-manifold M is one in which the n-simplexes σ_i^n have been given an ordering.

In the following we will always mean ordered simplicial n-manifold when we say simplificial n-manifold. We will use the initial data to associate vector spaces with ordered simplicial n-manifolds. Two such spaces whose arguments differ only by an ordering will easily be seen to be isomorphic. The maps Z will also be defined in such a way that up to these isomorphisms on the spaces V, they will be independent of the ordering of the simplicial manifolds.

Given a simplicial n-manifold M with $(n-1)$-colouring a, define

$$V(\sigma^n; a) \equiv V^{a_1 \cdots a_{n+1}}, \tag{2.3}$$

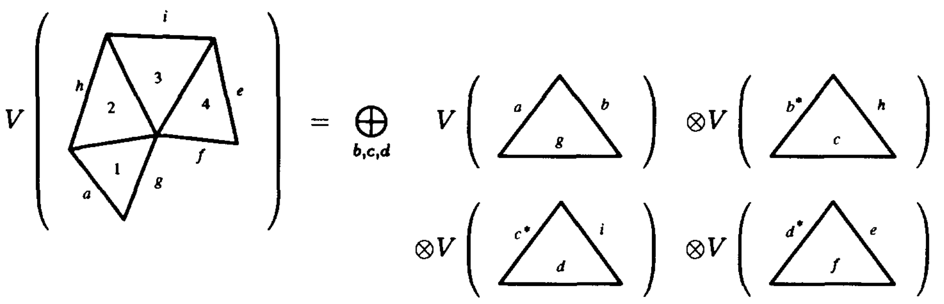
for each n-simplex $\sigma^n \in M$ where a_i or a_i^* are the colours of the $(n-1)$-simplexes, $\sigma_i^{n-1} \in (\sigma^n \cap \mathrm{M})$, depending upon whether σ^n comes before or after the n-simplex glued to it along σ_i^{n-1} in the ordering. (Note that the orientation of σ^n gives an ordering, up to even permutations, of the $\sigma_i^{n-1} \in (\sigma^n \cap S)$ which is also needed to specify the choice of $V^{a_1 \cdots a_{n+1}}$.)

Clearly a k-colouring of M induces a k-colouring of ∂M. With a simplicial n-manifold M and an $(n-1)$-colouring $\hat{a}$ of $\partial \mathrm{M}$ we associate the vector space

$$V(M; \hat{a}) \equiv \bigoplus_{a} \left(\bigotimes_{\sigma^n \in M} V(\sigma^n; a) \right), \tag{2.4}$$

where the sum is over all $(n-1)$-colourings a of M which induce $\hat{a}$ as an $(n-1)$-colouring of ∂M. The tensor product in (2.4) is in the order specified by the ordering of M (see Fig. 2).

For each of the spaces $V^{a_1 \cdots a_d}$ we choose a basis, which we denote



$$= \bigoplus_{b,c,d} (V^{abg} \otimes V^{b^*hc} \otimes V^{c^*id} \otimes V^{d^*ef})$$

Fig. 2. Constructing the vector space associated with an ordered simplicial 2-manifold whose boundary is coloured (a, h, i, e, f, g).

$[a_1 \cdots a_d]_\alpha \in V^{a_1 \cdots a_d}$, such that they satisfy

$$\langle [a_1 \cdots a_d]_\alpha, [b_1^* \cdots b_d^*]_\beta \rangle = \delta_{\alpha,\beta}$$

$$\text{where } b_1 \cdots b_d \text{ is an odd permutation of } a_1 \cdots a_d . \tag{2.5}$$

Definition 2.11. With respect to the above choice of basis we define a labelling α of an $(n-1)$-coloured simplicial n-manifold N to be a choice of basis vector at each n-simplex.

Let N be a simplicial n-manifold with $(n-1)$-colouring a and labelling α. We then denote

$$[a]_\alpha \equiv \bigotimes_{\sigma^n \in N} [a_1 \cdots a_n]_{\alpha_i} \in V(N; \hat{a}) , \tag{2.6}$$

where a is the $(n-1)$-colouring of ∂N induced from a. If $\partial N = 0$ these are the vector spaces $V(N)$ of the SQFT. Now consider the case where $N = \partial M$ for a simplicial d-manifold M so that $d = n+1$.

The maps $Z(M) : V(\partial M) \to \mathbb{C}$ are then defined in terms of $Z(\Delta^d)$ by

$$Z(M) \cdot [\hat{a}]_{\hat{\alpha}} \equiv \omega^{2\#_d(\mathrm{Int}\, M) + \#_d(\partial M)}$$

$$\times \prod_{\sigma_i^{d-2} \in \partial M} \omega_{\hat{a}_i} \sum_{a,\alpha} \left(\prod_{\sigma_j^{d-2} \in \mathrm{Int}\, M} \omega_{a_j}^2 \prod_{\sigma_k^d \in M} \hat{Z}(\Delta_i^d) \cdot [a_k]_{\alpha_h} \right), \tag{2.7}$$

with

$$\hat{Z}(\Delta_i^d) \cdot [a_k]_{\alpha_k} \equiv \omega^{-\#_d(\partial \Delta_h^d)} \prod_{\sigma_i^{d-2} \in \partial \Delta_h^d} \omega_{a_i}^{-1} Z(\Delta_k^d) \cdot [a_k]_{\alpha_h} , \tag{2.8}$$

where the following is implied:

(i) $\omega \in \mathbb{C}$ is an arbitrary parameter and

$$\#_d(C) \equiv \begin{cases} (-1)^d (C_0 - C_1 \cdots (-1)^{d-3} C_{d-3}) , & \text{for } d>2 , \\ 0 , & \text{otherwise} , \end{cases}$$

for any complex C where C_i is the number of i-simplexes in C.

(ii) $\hat{\alpha}$ and $\hat{a}$ are a labelling and $(d-2)$-colouring of the simplicial $(d-1)$-manifold ∂M.

(iii) The sum is over all $(d-2)$-colourings a and labellings α of M which induce $\hat{a}$ and $\hat{\alpha}$ on ∂M.

(iv) Δ_k^d is the simplex σ_k^d together with its faces.

(v) α_i and a_i are a labelling and $(d-2)$-colouring of Δ_i^d induced from α and a.

(vi) a_i is the colour of the $(d-2)$-simplex σ_i^{d-2}.

It is straightforward to check that these maps are multiplicative in the sense of SQFT axiom (S4).

2.2.2. Elementary moves and topological constraints on the initial data

A SQFT is said to be topological if $Z(M_1) = Z(M_2)$ whenever M_1 and M_2 have the same topology (i.e., $\exists$ a piecewise linear homeomorphism $f: M_1 \to M_2$) and ∂M_1 and ∂M_2 are isomorphic as simplicial complexes (i.e., $\exists$ a simplicial isomorphism between them).

Note that this is equivalent to the definition given by Atiyah in Ref. [1], where the maps $Z(\mathcal{M})$ and vector spaces $V(\mathcal{S})$ are associated with d-manifolds $\mathcal{M}$ and their boundaries $\mathcal{S}$. Following Ref. [12] one may construct a vector space $V(\mathcal{S})$ associated with a closed $(d-1)$-manifold $\mathcal{S}$ using a simplicial d-manifold with the topology of $\mathcal{S} \times \mathcal{B}^1$, where $\mathcal{B}^1$ denotes a one-dimensional ball. Given two closed simplicial $(d-1)$-manifolds S_1 and S_2 in the equivalence class $\mathcal{S}$, let C_{S_1,S_2} be such a cylinder simplicial d-manifold with $\partial C_{S_1,S_2} = S_1 \cup S_2^*$. Then define the map $\mathrm{Cyl}_{S_1,S_2} : V(S_1) \to V(S_2)$ by

$$Z(C_{S_1,S_2}) \cdot (V(S_1) \otimes V(S_2^*)) = \langle \mathrm{Cyl}_{S_1,S_2} \cdot V(S_1), V(S_2^*) \rangle . \tag{2.9}$$

We can now define $V(\mathcal{S}) \equiv V(S_1)/\sim$ where $\sim$ is the equivalence relation induced by the map Cyl_{S_1,S_2}. Clearly, $Z(M)$ induces a map $Z(\mathcal{M}) : V(\partial\mathcal{M}) \to \mathbb{C}$ for each d-manifold $\mathcal{M}$.

When the SQFT is constructed from the abstract data given above, the topological constraint on Z enforces strong conditions on the initial data. As the initial data is local in the simplicial manifold we can express these conditions directly on the initial data if we can express a general piecewise linear homeomorphism on M in terms of a product of local maps, i.e. maps which are equal to the identity outside a d-ball neighbourhood of a point in M. Many such theorems exist, the first being due to Alexander [2] in the 30's. The following theorem due to Pachner [3] is the most suited for our purposes as it expresses a general piecewise linear homeomorphism in terms of the product of a finite number of local

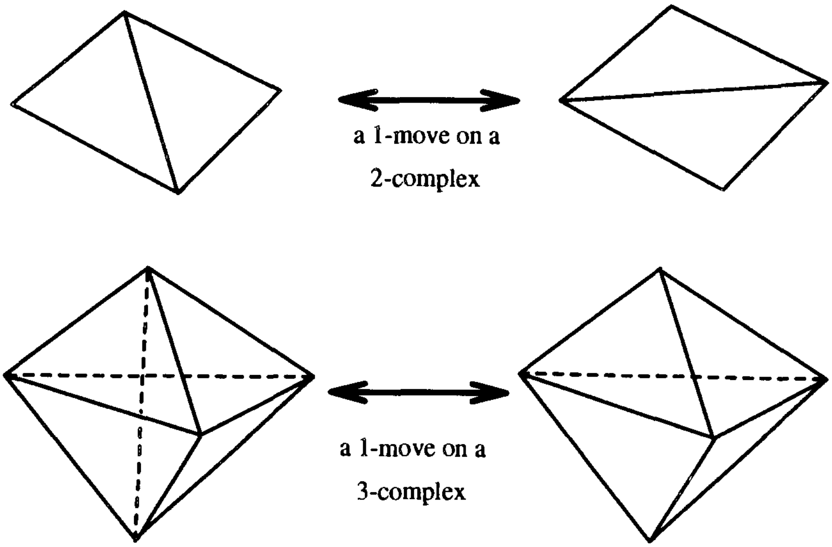


Fig. 3. Examples of elementary moves.

piecewise linear homeomorphisms, or "moves" on the triangulation. To describe these moves we need the following definitions.

Definition 2.12. The join of two simplicial complexes K and L is the simplicial complex

$$K \bullet L \equiv \{(x_0, ..., x_n, y_0, ..., y_m) : (x_0, ..., x_n) \in K \quad \text{and} \quad (y_0, ..., y_m) \in L\} .$$

Definition 2.13. An elementary k-move on a simplicial n-manifold M is the replacing of a subcomplex $\Delta^k \bullet \partial \Delta^{n-k}$ of M by a complex $\partial \Delta^k \bullet \Delta^{n-k}$, which has identical boundary (see Fig. 3 for examples).

The central theorem and corollary.

Theorem. (Pachner [3]). *If ∂M_1 is isomorphic to ∂M_2 then $|M_1|$ and $|M_2|$ are homeomorphic if and only if M_2 is the result of a finite number of elementary moves on a simplicial manifold isomorphic to M_1.*

We may now state the corollary of this theorem which is central to our construction (this result was first pointed out in Ref. [8]).

Corollary. *The initial data I, $*$, ω_a, $V^{a_1 \cdots a_d}$ and $Z(\Delta^d)$ define a TQFT via the above construction if $Z(\Delta^d)$ is such that the following map identities hold:*

$$Z(\Delta^k \bullet \partial\Delta^{d-k}) = Z(\partial\Delta^k \bullet \Delta^{d-k}) , \quad \textit{for } 0 \leqslant k \leqslant d , \tag{2.10}$$

where $Z(M)$ is defined in terms of $Z(\Delta^d)$ by Eq. (2.7), with

$$\omega^2 \equiv \sum_{a \in I} \omega_a^4 . \tag{2.11}$$

2.3. The dual cell decomposition and coloured d-graph invariants

In this section we translate the construction to the cell decomposition dual to the simplicial manifold. The identities (2.10) will be seen to be equivalent to certain identities satisfied by topological invariants of graphs embedded in a $(d-1)$-dimensional sphere. We must first run through some more definitions.

Definition 2.14. The barycentre of an n-simplex σ^n, denoted $\mathrm{b}(\sigma^n)$, is the point in $|\sigma^n|$, as defined in definition 2.1, where each $\lambda_i = 1/(n+1)$.

Definition 2.15. With each sequence of simplexes $\sigma^{p_1} \subset \sigma^{p_2} \subset \cdots \subset \sigma^{p_k}$ with $p_i < p_{i+1}$, we associate a k-simplex $(\mathrm{b}(\sigma^{p_1}), ..., \mathrm{b}(\sigma^{p_k}))$ whose vertices are the barycentres of $\sigma^{p_1}, ..., \sigma^{p_k}$. The barycentric star, $\mathrm{bstar}(\sigma^n)$, of an n-simplex σ^n in a simplicial complex K is the simplicial complex

$$\mathrm{bstar}(\sigma^n) = \{(\mathrm{b}(\sigma^n), \mathrm{b}(\sigma^{p_1}), ..., \mathrm{b}(\sigma^{p_k})) : \sigma^{p_i} \in K\} . \tag{2.12}$$

Definition 2.16. The barycentric subdivision of a simplicial manifold M is the set $\{\mathrm{bstar}(\sigma) : \sigma \in M\}$ of simplicial complexes (see Fig. 4 for examples).

Definition 2.17. The n-strata of the barycentric subdivision of a simplicial d-manifold M is the space

$$\bigcup_{(\sigma^{d-n-1}) \in M} |\partial(\mathrm{bstar}(\sigma^{d-n-1}))| . \tag{2.13}$$

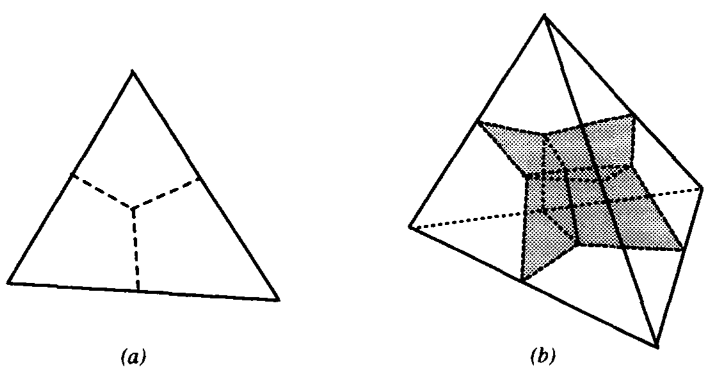


Fig. 4. The barycentric subdivision of (a) a simplicial 2-manifold and (b) a simplicial 3-manifold.

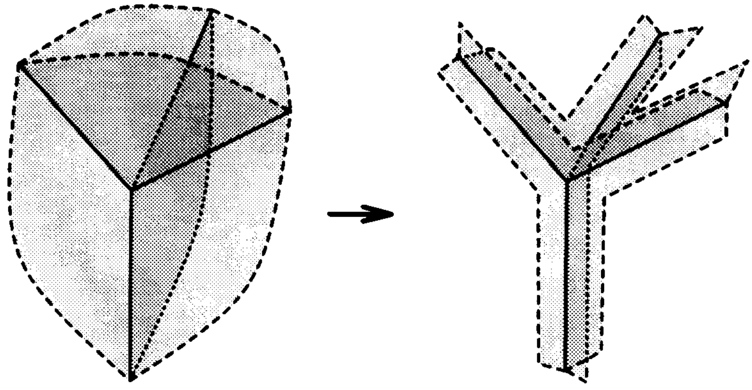


Fig. 5. A simple 2-polyhedron and its associated 4-graph $\mathcal{I}_{\mathcal{B}^3}$ embedded in $\mathcal{B}^3$.

Definition 2.18. A simple n-polyhedron is a topological space in which each point has a neighbourhood homeomorphic to the neighbourhood of some point in the n-strata of the barycentric subdivision of Δ^{n+1} (see Fig. 5).

Definition 2.19. An n-graph $\mathcal{G}$ is a neighbourhood of the 1-strata in a simple $(n-2)$-polyhedron.

For a given colour set I we define a colouring a of an n-graph $\mathcal{G}$ to be a map from the links of $\mathcal{G}$ to I. By an ordered n-graph we mean an n-graph together with an ordering of its vertices. However, as we did for simplicial n-manifolds in Section 2.2 we will now be slightly loose with our language and will always imply ordered n-graph when we say n-graph. We now consider embeddings of coloured d-graphs $\mathcal{G}$ in orientable $(d-1)$-manifolds $\mathcal{M}$, and denote such an embedding by $\mathcal{G}_{\mathcal{M}}$. As we did for simplicial d-manifolds we associate the initial data $V(\rho, a) \equiv V^{a_1,\dots,a_n}$ with each n-valent vertex ρ, where a_i or a_i^* are the colours of the links meeting at ρ. The ordering of the vertices clearly induces an orientation on each link which we use to make this choice. (Note that it is the orientation of $\mathcal{M}$ that supplies an ordering up to even permutations of the links meeting at ρ, which is also needed in order to specify $V^{a_1,\dots,a_n}$.) We may now associate the space

$$V(\mathcal{G}_{\mathcal{M}}; \hat{a}) \equiv \bigoplus_{a} \left(\bigotimes_{\rho \in \mathcal{G}} V(\rho, a) \right), \tag{2.14}$$

with an embedded graph $\mathcal{G}_{\mathcal{M}}$ whose boundary links (those containing only one n-valent vertex) are coloured $\hat{a}$. In (2.14) the sum is over all colourings a which induce $\hat{a}$.

With an embedding, $\mathcal{G}_{\mathcal{S}^{d-1}}$, of a closed d-graph $\mathcal{G}$ in the $(d-1)$-dimensional sphere $\mathcal{S}^{d-1}$ we associate maps $Z(\mathcal{G}_{\mathcal{S}^{d-1}}) : V(\mathcal{G}_{\mathcal{S}^{d-1}}) \to \mathbb{C}$. (Note that we write

$V(\mathcal{G}) \equiv V(\mathcal{G}; \hat{a})$ when $\mathcal{G}$ is closed.) In order to describe how the identities (2.10) on the initial data translate to map identities between these maps $Z(\mathcal{G}_{\mathcal{S}^{d-1}})$, we will require the following standard graph embeddings.

(i) Denote by $\mathcal{I}_{\mathcal{B}^{d-1}}$ (respectively $\mathcal{P}_{\mathcal{B}^{d-1}}$) the open d-graph containing only one vertex (respectively two vertices) and embedded in the $(d-1)$-dimensional ball $\mathcal{B}^{d-1}$ in such a way that it is homeomorphic to the neighbourhood of the 1-strata in the $(d-2)$-strata of the barycentric subdivision of Δ^{d-1} (respectively $\Delta^{d-1} \cup \Delta^{d-1}$). Here $\Delta^{d-1} \cup \Delta^{d-1}$ denotes the simplicial manifold made from gluing two copies of Δ^{d-1} along a common $(d-2)$-simplex in their boundaries. Figs. 5 and 6 show examples.

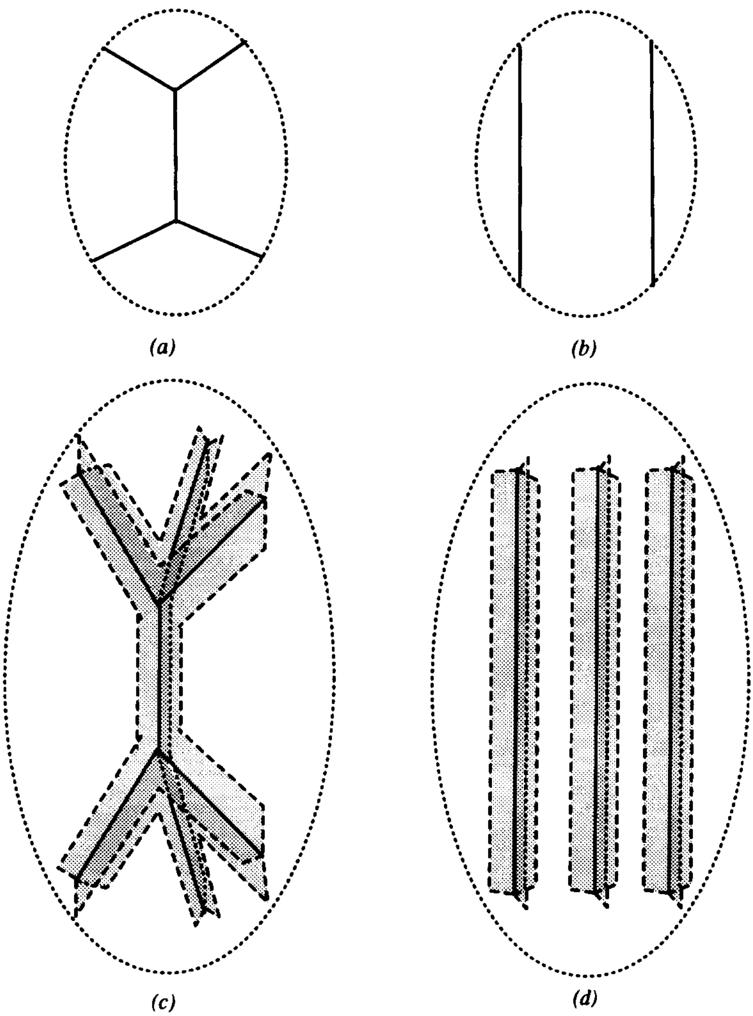


Fig. 6. Some standard d-graph embeddings, (a) $\mathcal{P}_{\mathcal{B}^2}$, (b) $\mathcal{Q}_{\mathcal{B}^2}$, (c) $\mathcal{P}_{\mathcal{B}^3}$, and (d) $\mathcal{Q}_{\mathcal{B}^3}$.

(ii) Denote by $\mathcal{Q}_{B^{d-1}}$ the d-graph embedding consisting of $(d-1)$ parallel strands in $\mathcal{B}^{d-1}$ (see Fig. 6).

(iii) Denote by $\mathcal{D}_{\mathcal{S}^{d-1}}$ the closed d-graph with $(d+1)$ vertices, embedded in $\mathcal{S}^{d-1}$ in such a way that it is homeomorphic to the neighbourhood of the 1-strata in the $(d-2)$-strata of the barycentric subdivision of $\partial\Delta^d$.

Theorem. *The initial data I, $*$, ω_a and $V^{a_1\cdots a_d}$ together with the maps $Z(\mathcal{G}_{\mathcal{S}^{d-1}})$ satisfying the following two identities (2.15) and (2.16), provide the initial data for a TQFT.*

The first identity corresponds to fusing two d-graph embeddings into one,

$$\sum_\alpha Z(\mathcal{X}_{\mathcal{B}^{d-1}} \cup \mathcal{I}_{\mathcal{B}^{d-1}}) \cdot (x(\mathbf{a}^*) \otimes [a_1 \cdots a_d]_\alpha)$$
$$\times Z(\mathcal{Y}_{\mathcal{B}^{d-1}} \cup \mathcal{I}_{B^{d-1}}) \cdot (y(\mathbf{a}) \otimes [a_2^* a_1^* a_3^* \cdots a_d^*]_\alpha)$$
$$= Z(\mathcal{X}_{\mathcal{B}^{d-1}} \cup \mathcal{Y}_{\mathcal{B}^{d-1}}) \cdot (x(\mathbf{a}^*) \otimes y(\mathbf{a})) \,, \tag{2.15}$$

where $\mathcal{X}_{\mathcal{B}^{d-1}}$ and $\mathcal{Y}_{\mathcal{B}^{d-1}}$ are d-graphs each with d boundary links so that $\mathcal{X}_{\mathcal{B}^{d-1}} \cup \mathcal{I}_{\mathcal{B}^{d-1}}$ and $\mathcal{Y}_{\mathcal{B}^{d-1}} \cup \mathcal{I}_{\mathcal{B}^{d-1}}$ are embeddings of closed graphs in $\mathcal{S}^{d-1}$, $x(\mathbf{a}^*) \in V(\mathcal{X}_{\mathcal{B}^{d-1}}; \mathbf{a}^*)$ so that $x(\mathbf{a}^*) \otimes [a_1 \cdots a_d]_\alpha \in V(\mathcal{X}_{\mathcal{B}^{d-1}} \cup \mathcal{I}_{\mathcal{B}^{d-1}})$, and similarly for $y(\mathbf{a}) \in V(\mathcal{Y}_{\mathcal{B}^{d-1}})$.

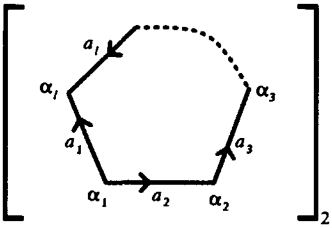
The second identity corresponds to removing two adjacent d-graph vertices,

$$\sum_{a_1,\alpha} \omega_{a_1}^2 Z(\mathcal{T}_{\mathcal{B}^{d-1}} \cup \mathcal{P}_{\mathcal{B}^{d-1}}) \cdot (t(a_2 \cdots a_d, a_2^* \cdots a_d^*) \otimes [a_1 \cdots a_d]_\alpha \otimes [a_2^* a_1^* a_3^* \cdots a_d^*]_\alpha)$$
$$= Z(\mathcal{T}_{\mathcal{B}^{d-1}} \cup \mathcal{Q}_{\mathcal{B}^{d-1}}) \cdot t(a_2 \cdots a_d, a_2^* \cdots a_d^*) \,, \tag{2.16}$$

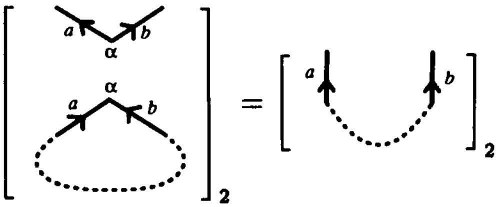
where $\mathcal{T}_{\mathcal{B}^{d-1}}$ is a d-graph with $2(d-1)$ boundary links so that $\mathcal{T}_{\mathcal{B}^{d-1}} \cup \mathcal{P}_{\mathcal{B}^{d-1}}$ is the embedding of a closed graph in $\mathcal{S}^{d-1}$, and $t(a_2 \cdots a_d, a_2^* \cdots a_d^*) \in V(\mathcal{T}_{\mathcal{B}^{d-1}}; \hat{\mathbf{a}})$, with $\hat{\mathbf{a}} = (a_2, \ldots, a_d, a_2^*, \ldots, a_d^*)$.

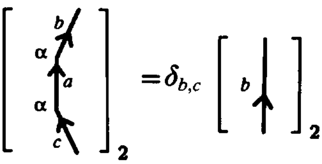
Proof. We first note that $V(\mathcal{D}_{\mathcal{S}^{d-1}}) = V(\partial\Delta^d)$, as is clear from Eqs. (2.4) and (2.14). Hence we may make the identification $\hat{Z}(\Delta^d) \equiv Z(\mathcal{D}_{\mathcal{S}^{d-1}})$. We now show using diagrammatic techniques for the three cases $d=2$, 3 and 4 that the $Z(M)$ constructed via Eq. (2.7) using the identities (2.15) and (2.16) together with $\omega^2 \equiv \sum_{a\in I} \omega_a^4$, satisfy the identities (2.10) and hence give the required result.

The 2-graph identities. Denote by $[\mathbf{a}]_\alpha \equiv [a_1 \cdots a_l]_{\alpha_1 \cdots \alpha_l} \equiv [a_1, a_2]_{\alpha_1} \otimes [a_2^*, a_3]_{\alpha_2} \otimes [a_3^*, a_4]_{\alpha_3} \cdots \otimes [a_l^*, a_1^*]_{\alpha_l}$ the basis of $V(\mathcal{G}_{\mathcal{S}^1})$, where $\mathcal{G}$ is a closed 2-graph with l vertices. We denote the complex numbers $Z(\mathcal{G}_{\mathcal{S}^1}) \cdot [\mathbf{a}]_\alpha$ by the following 2-graph diagram:

$Z(\mathcal{G}_{\mathcal{S}^1})\cdot[a]_\alpha \equiv$  . (2.17)

The conditions (2.15) and (2.16) on Z are then represented by the following 2-graph identities:

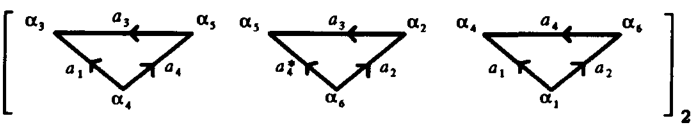
$\sum_\alpha$  , (2.18)

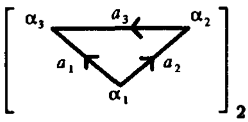
$\sum_{\alpha,a} \omega_a^2$  , (2.19)

where, as is standard practice in such notation, one means that outside the section of the diagram that is shown in the bracket the diagrams are equivalent.

Denote by $\alpha\equiv(\alpha_1, ..., \alpha_6)$ and $a=(a_1, ..., a_4)$ the labelling and 0-colouring of $\Delta^0\bullet\partial\Delta^2$ shown in Fig. 7. This clearly induces $\hat{\alpha}\equiv(\alpha_1,\alpha_2,\alpha_3)$ and $\hat{a}$ on $\partial(\Delta^0\bullet\partial\Delta^2)=\partial\Delta^0\bullet\partial\Delta^2$. Then for each basis element $[a_1a_2a_3]_{\alpha_1\alpha_2\alpha_3}$ of $V(\partial\Delta^0\bullet\partial\Delta^2;\hat{a})$ we have

$$Z(\Delta^0\bullet\partial\Delta^2)\cdot[a_1a_2a_3]_{\alpha_1\alpha_2\alpha_3}$$
$$=\omega_{a_1}\omega_{a_2}\omega_{a_3}\sum_{a_4,\alpha_4,\alpha_5,\alpha_6}\omega_{a_4}^2$$
$$\times\hat{Z}(\Delta^2)\cdot[a_1a_4a_3]_{\alpha_4\alpha_5\alpha_3}\hat{Z}(\Delta^2)\cdot[a_4^*a_2a_3]_{\alpha_6\alpha_2\alpha_5}\hat{Z}(\Delta^2)\cdot[a_1a_2a_4]_{\alpha_1\alpha_6\alpha_4}$$
$$=\omega_{a_1}\omega_{a_2}\omega_{a_3}\sum_{a_4,\alpha_4,\alpha_5,\alpha_6}\omega_{a_4}^2$$

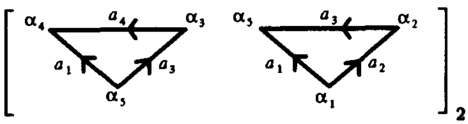
$\times$ 

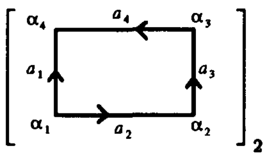
$= \omega_{a_1}\omega_{a_2}\omega_{a_3}$ 

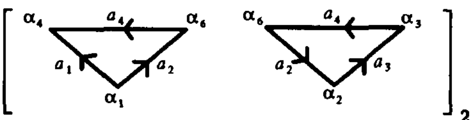
$$= Z(\partial\Delta^0 \bullet \Delta^2)\cdot[a_1 a_2 a_3]_{\alpha_1\alpha_2\alpha_3}\,, \qquad (2.20)$$

proving invariance under elementary 0-moves. Note that we have used the SQFT axiom (S4), which says that Z is multiplicative for disjoint manifolds, as well as Eqs. (2.18) and (2.19). We prove the invariance of Z under an elementary 1-move in a similar way:

$$Z(\Delta^1 \bullet \partial\Delta^1)\cdot[a_1 a_2 a_3 a_4]_{\alpha_1\alpha_2\alpha_3\alpha_4}$$

$= \omega_{a_1}\cdots\omega_{a_4}\sum_{\alpha_5}$ 

$= \omega_{a_1}\cdots\omega_{a_4}$ 

$= \omega_{a_1}\cdots\omega_{a_4}\sum_{\alpha_6}$ 

$$= Z(\partial\Delta^1 \bullet \Delta^1)\cdot[a_1 a_2 a_3 a_4]_{\alpha_1\alpha_2\alpha_3\alpha_4}\,. \qquad (2.21)$$

The 3-graph identities. Here $\mathscr{G}$ is a closed 3-graph and for the basis $[a]_\alpha$ of $V(\mathscr{G}_{\mathscr{S}^2})$ we represent $Z(\mathscr{G}_{\mathscr{S}^2})\cdot[a]_\alpha$ by a coloured and labelled 3-valent graph with oriented links, drawn on the plane as in Fig. 8. The $\mathbb{C}$-numbers that these

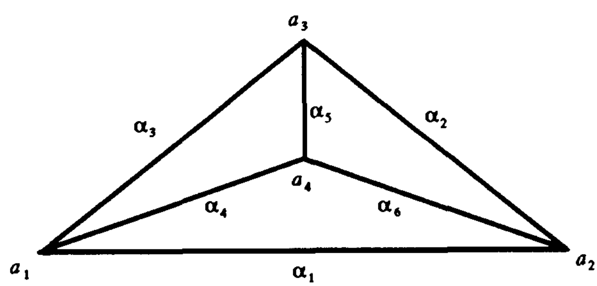


Fig. 7. A 0-colouring and labelling of $\Delta^0 \bullet \partial\Delta^2$.

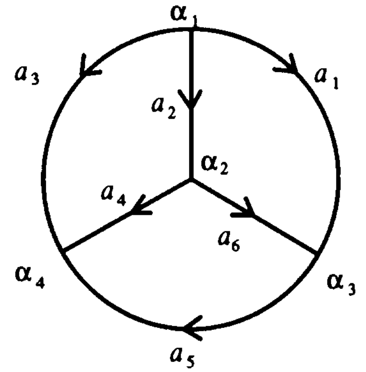
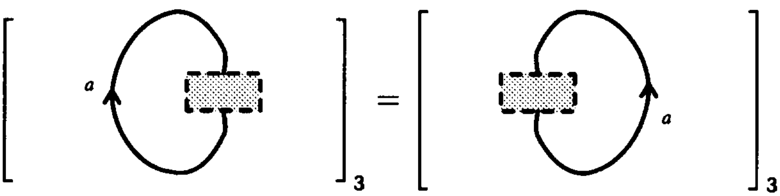


Fig. 8. A coloured and labelled 3-graph diagram.

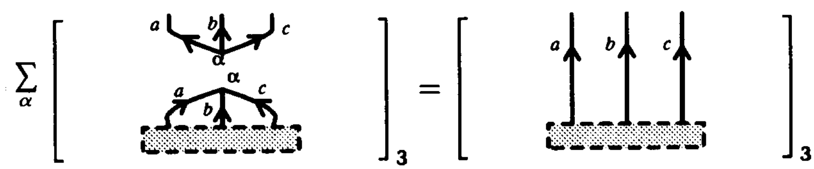
diagrams represent are invariant under homeomorphisms of the diagrams embedding in the plane, as well as being invariant under the following move due to the fact that $\mathcal{G}$ is embedded in $\mathcal{S}^2$ and not $\mathbb{R}^2$:



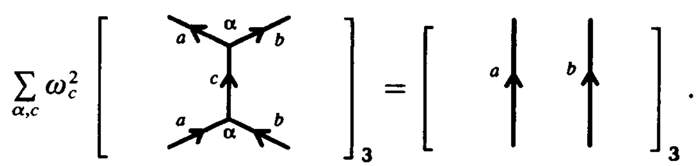
(2.22)

Here the shaded box represents an arbitrary diagram.

The conditions (2.15) and (2.16) on Z are represented by the following identities:



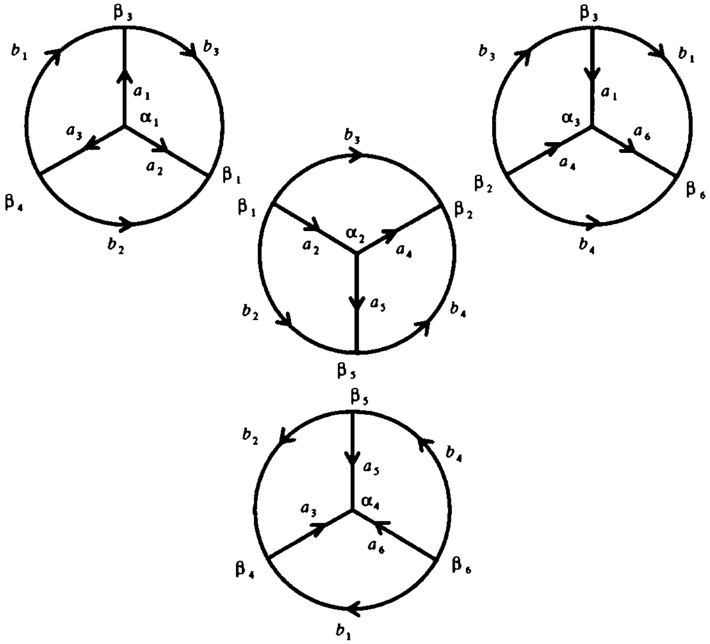
, (2.23)

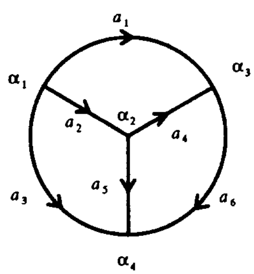


(2.24)

Using (2.22), (2.23) and (2.24) we can show that $Z(M)$ is invariant under elementary k-moves. Firstly for a 0-move

$$Z(\Delta^0 \bullet \partial\Delta^3)\cdot[a_1\cdots a_6]_{\alpha_1\cdots\alpha_4} = \omega^{-6}\omega_{a_1}\cdots\omega_{a_6} \sum_{\beta_1\cdots\beta_6, b_1\cdots b_4} \omega^2_{b_1}\cdots\omega^2_{b_4}$$

$\times \Big[$  $\Big]_3$

$= \omega^{-4}\omega_{a_1}\cdots\omega_{a_6} \Big[$  $\Big]_3$

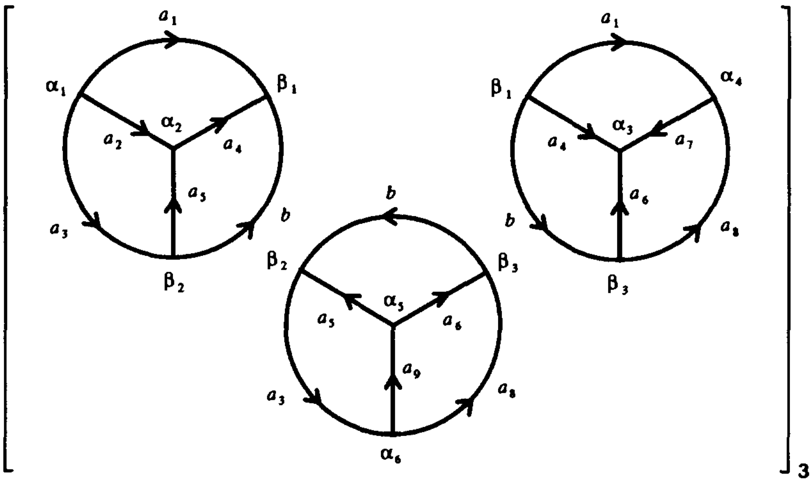
$$= Z(\partial\Delta^0 \bullet \Delta^3)\cdot[a_1\cdots a_6]_{\alpha_1\cdots\alpha_4}, \tag{2.25}$$

on using (2.23) and (2.24).

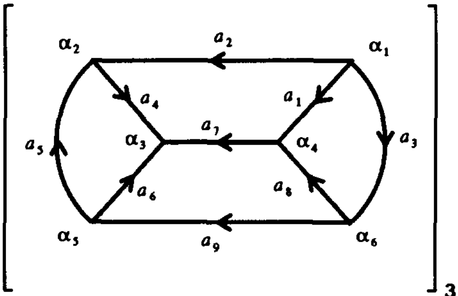
Similarly for an elementary 1-move

$$Z(\Delta^1 \bullet \partial\Delta^2)\cdot[a_1\cdots a_9]_{\alpha_1\cdots\alpha_6} = \omega^{-5}\omega_{a_1}\cdots\omega_{a_9}\sum_{\beta_1\beta_2\beta_3,b}\omega_b^2$$

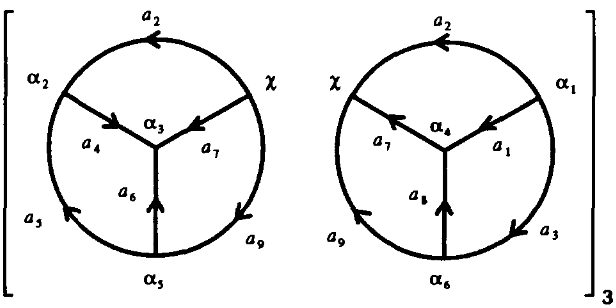
$\times$



$= \omega^{-5}\omega_{a_1}\cdots\omega_{a_9}$



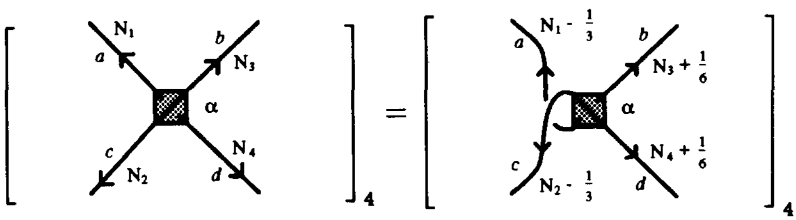
$= \sum_{\chi}\omega^{-5}\omega_{a_1}\cdots\omega_{a_9}$



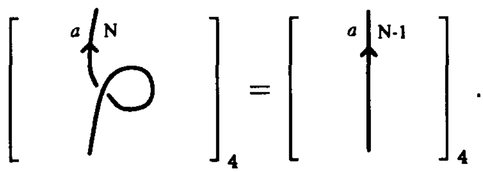
$$= Z(\partial\Delta^1 \bullet \Delta^2)\cdot[a_1\cdots a_9]_{\alpha_1\cdots\alpha_6}. \qquad (2.26)$$

The 4-graph identities. Here $\mathcal{G}_{\mathcal{S}^3}$ is an embedding of a 4-graph $\mathcal{G}$ in $\mathcal{S}^3$. We represent the $\mathbb{C}$-numbers $Z(\mathcal{G}_{\mathcal{S}^3})\cdot[a]_\alpha$ by what we call coloured and labelled 4-graph diagrams. These are knot diagrams with the 4-valent vertices drawn as in

Eq. (2.27), together with a map from its links to the integers $\mathbb{Z}$. (Note that this map is denoted on the diagram as multiples of $\frac{1}{6}$ for convenience ($\frac{6}{6}$ being a twist).) The $\mathbb{C}$-numbers that these diagrams represent are invariant under the standard Reidemeister moves of regular isotopy in $\mathbb{R}^3$, together with the following two moves:

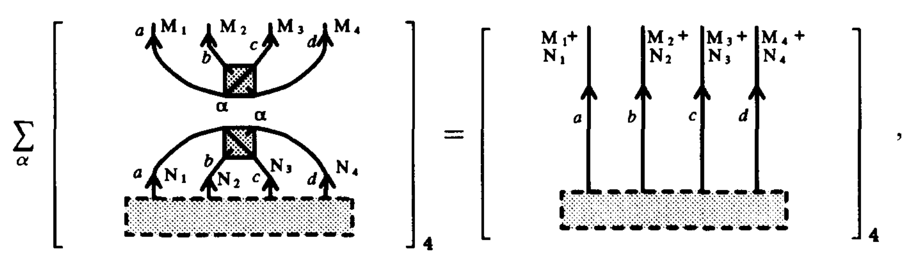


(2.27)

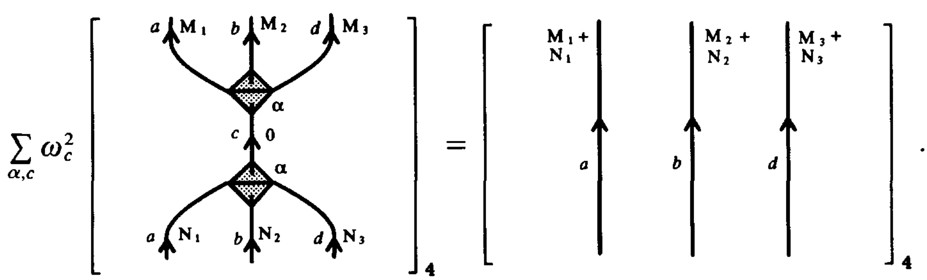


(2.28)

The conditions (2.15) and (2.16) on Z are represented by the following 4-graph identities:



(2.29)

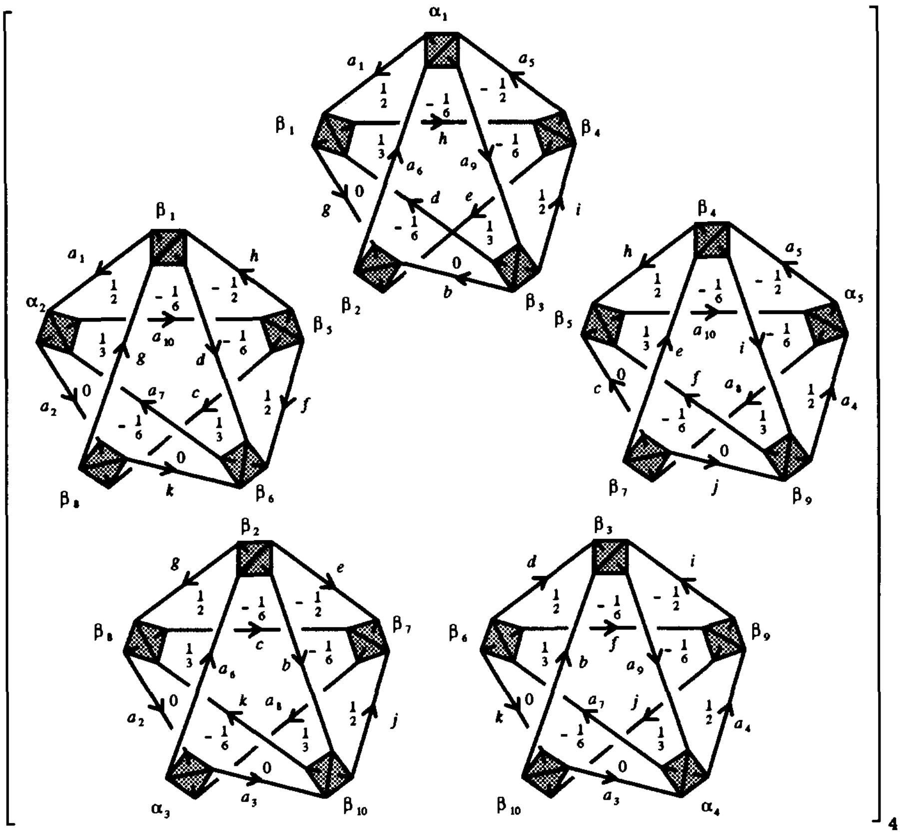


(2.30)

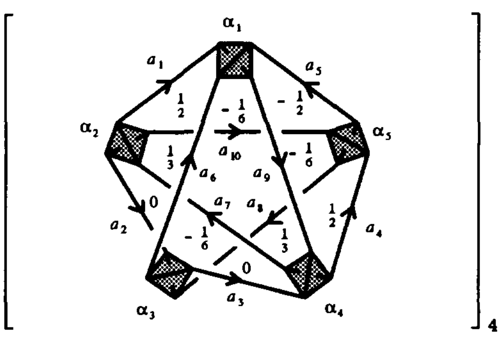
The invariance of $Z(M)$ under elementary 0-, 1-, and 2-moves can be shown by using (2.27), (2.28), (2.29) and (2.30), together with regular isotopy Reidemeister moves.

$$Z(\Delta^0 \bullet \partial\Delta^4)\cdot[a_1\cdots a_{10}]_{\alpha_1\cdots\alpha_5} = \omega^{-13}\omega_{a_1\cdots\omega_{a_{10}}} \sum_{\beta_1\cdots\beta_{10},b\cdots k} \omega_b^2\cdots\omega_k^2$$

$\times$



$$= \omega^{-5}\omega_{a_1}\cdots\omega_{a_{10}}$$



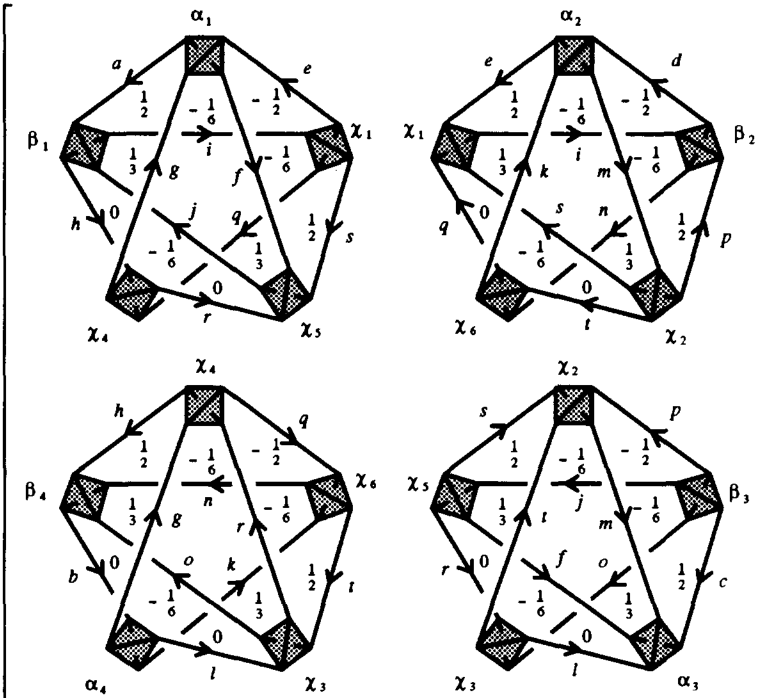
$$= Z(\partial\Delta^0 \bullet \Delta^4)\cdot[a_1\cdots a_{10}]_{\alpha_1\cdots\alpha_5}\,. \tag{2.31}$$

Note that we have basically used (2.29) repeatedly to fuse the five disjoint 4-

graphs into one, and then used (2.30) to remove the extra vertices. We proceed similarly for the 1-move and 2-move

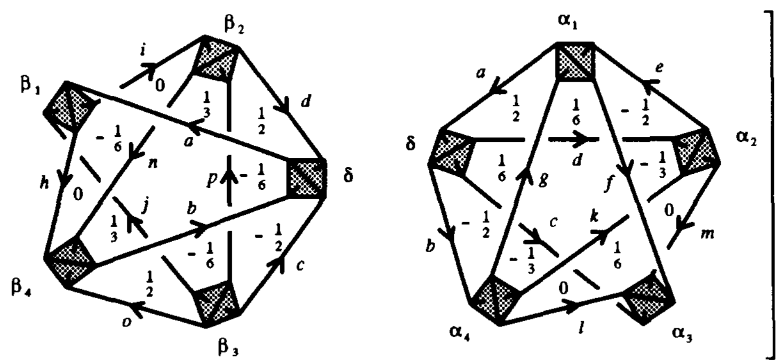
$$Z(\Delta^1 \bullet \partial \Delta^3)\cdot[a\cdots p]_{\alpha_1\cdots\alpha_4,\beta_1\cdots\beta_4} = \omega^{-10}\omega_a\cdots\omega_p \sum_{\chi_1\cdots\chi_6, q\cdots t} \omega_q^2\cdots\omega_t^2$$

$$\times \Bigg[\qquad \Bigg]_4$$



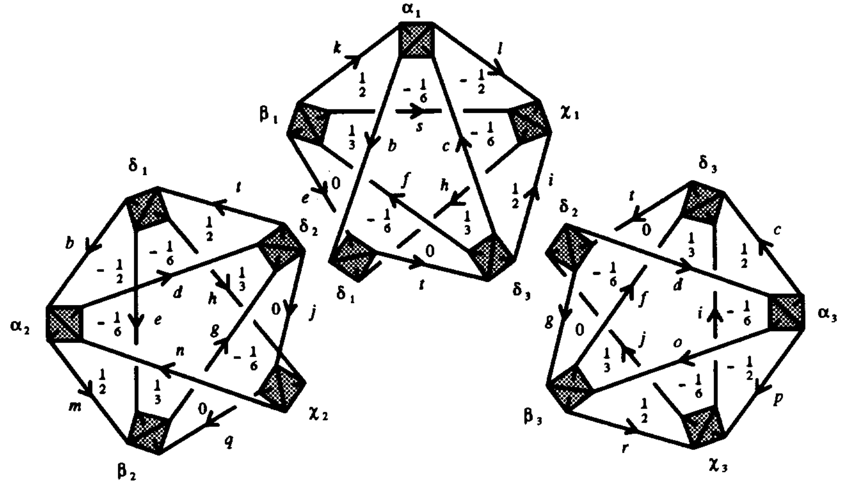
$$= \omega^{-8}\omega_a\cdots\omega_p$$

$$\times \sum_{\delta} \Bigg[\qquad \Bigg]_4$$



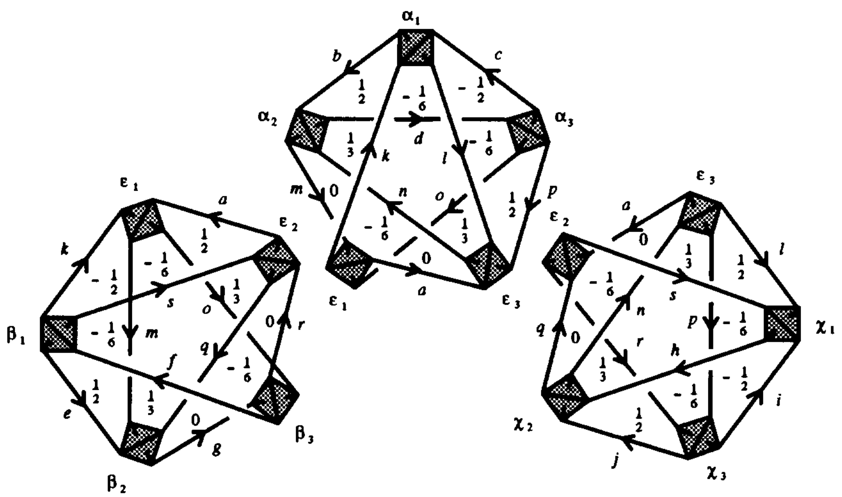
$$= Z(\partial\Delta^1 \bullet \Delta^3)\cdot[a\cdots p]_{\alpha_1\cdots\alpha_4,\beta_1\cdots\beta_4}\,, \tag{2.32}$$

$$Z(\Delta^2 \bullet \partial\Delta^2)\cdot[b\cdots s]_{\alpha,\beta,\chi} = \omega^{-3}\omega_b\cdots\omega_s \sum_{\delta_1\delta_2\delta_3,t} \omega_t^2$$

$\times$ 

$_4$

$$= \omega^{-3}\omega_b\cdots\omega_s \sum_a \omega_a^2$$

$\times$ 

$_4$

$$= Z(\partial\Delta^2 \bullet \Delta^2)\cdot[a\cdots p]_{\alpha_1\cdots\alpha_4,\beta_1\cdots\beta_4}\,. \tag{2.33}$$

3. Realisations from the Kauffman bracket polynomial

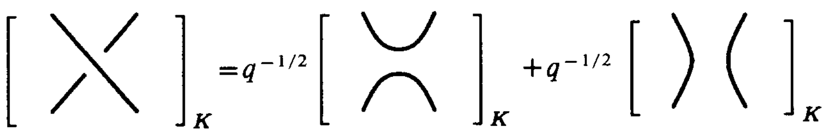
In this section we show how one may obtain initial data satisfying the $d \leqslant 4$

construction conditions of Section 2 directly from the Kauffman bracket polynomial in (q, q^{-1}), when q is a complex root of unity. It is clear from the construction that we require knot invariants with specific properties in order to define 4-graph values satisfying the identities (2.29) and (2.30), and hence the $d=4$ initial data for the TQFT. Such 4-graph values would induce 2- and 3-graph values which would define two- and three-dimensional TQFTs as well (this just being an example of the fact that d-dimensional TQFTs induce TQFTs of lower dimension). We will show explicitly how to construct 2-, 3-, and 4-graph values from the Kauffman bracket polynomial.

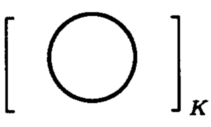
In three dimensions the model corresponds to the Kauffman–Lins model of Ref. [13] and hence the Turaev–Viro state-sum model of Ref. [12]. The q-deformed spin-networks that we build up from the Kauffman bracket polynomial are intimately related to the representation theory of the quantum group $U_q sl(2)$, in fact they are the recoupling theory. We will see that just as quantum $6j$-symbols play a leading role as the initial data $\hat{Z}(\Delta^3)\cdot[a]_\alpha$ in the three-dimensional model, it is certain quantum $15j$-symbols which form the initial data $\hat{Z}(\Delta^4)\cdot[a]_\alpha$ for the four-dimensional model.

3.1. *From q-deformed spin-networks to TQFTs*

The simplest model is based on the Kauffman bracket polynomial [14]. This associates a polynomial in q and q^{-1} with real coefficients, with each knot diagram and is an invariant of regular isotopy, which is the equivalence relation on knot diagrams generated by the Reidemeister moves (a), (b), (d) and (e) of Fig. 9. It is defined by the two relations



(3.1)



$$= -(q+q^{-1})\ . \qquad (3.2)$$

Reidemeister showed [15] that knot diagrams of any two embeddings of closed loops in $\mathbb{R}^3$ which can be continuously deformed into each other (ambient isotopy) are related by a sequence of the moves (b), (c), (d) and (e) of Fig. 9. The regular isotopy moves relate topologically equivalent embeddings of ribbons. The Jones polynomial [16] $J(\mathscr{L})$ of a knot $\mathscr{L}$, which is an ambient isotopy invariant, is related to the Kauffman bracket polynomial by the following relation:

$$J(\mathscr{L}) = (-q^{3/2})^{w(\mathscr{L})}[\mathscr{L}]_K\,, \qquad (3.3)$$

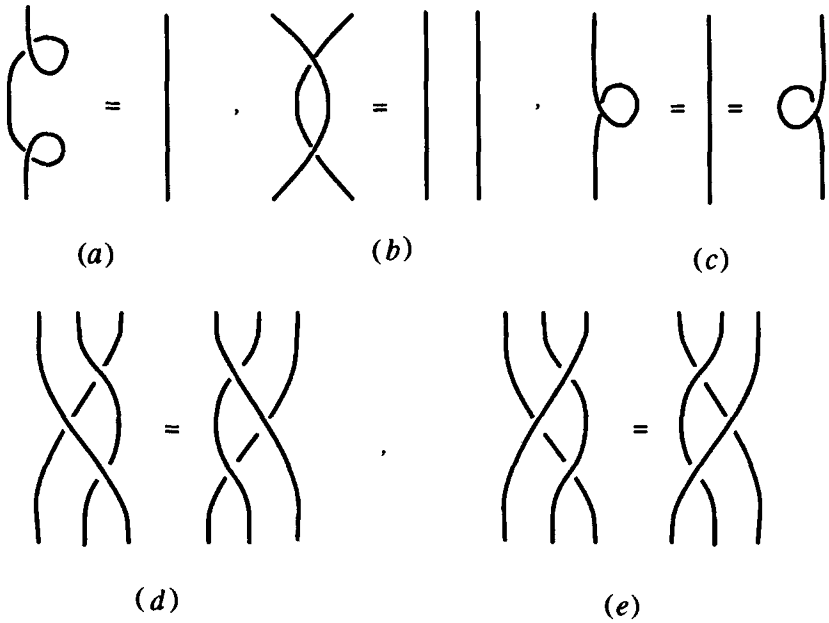
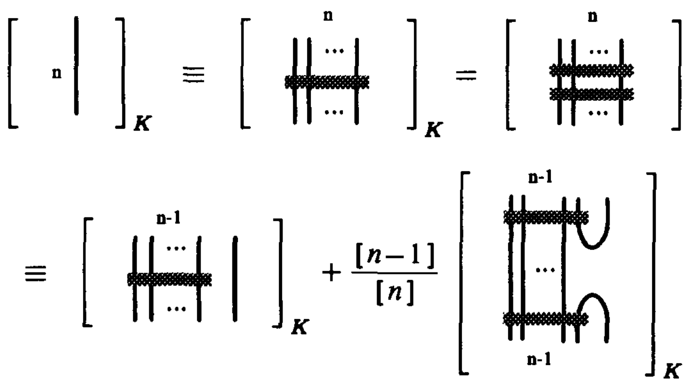


Fig. 9. The Reidemeister moves of ambient isotopy are (b), (c), (d) and (e), and those of regular isotopy are (a), (b), (d) and (e).

where $w(\mathscr{L})$ is the writhe, or number of twists in the ribbon. Note that (3.1) and (3.2) are sufficient to define the invariant as one can use (3.1) to reduce any knot diagram to a collection of unknots (3.2) (the bracket of a knot being defined such that it is the product of the brackets of the disjoint components).

The Kauffman bracket may be generalised to a regular isotopy invariant of integer coloured 3-valent graphs (q-deformed spin-networks) in the following way. First one defines coloured links by the linear combination of strands given iteratively by

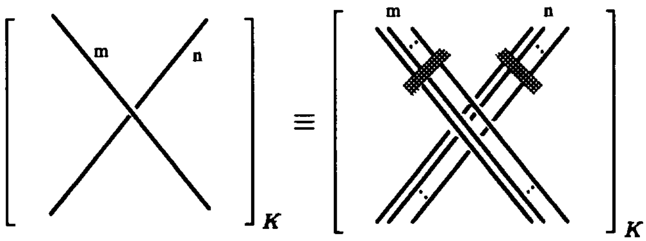


(3.4)

where $[n]$ is defined to be the following function of q:

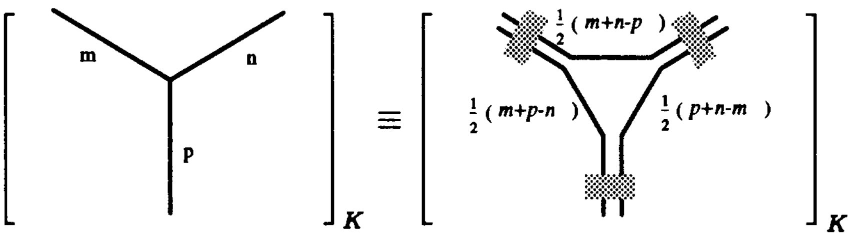
$$[n] \equiv (q^n - q^{-n})/(q - q^{-1}) . \tag{3.5}$$

Crossings are defined in the natural fashion by



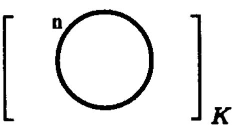
, (3.6)

and 3-valent vertices by



. (3.7)

It is clear that Eq. (3.7) may only be used to define a 3-valent vertex if each of $\frac{1}{2}(m+n-p)$, $\frac{1}{2}(p+m-n)$ and $\frac{1}{2}(m+p-n)$ is a non-negative integer. If the triple (m, n, p) satisfies this property we say that it is an admissible triple. The bracket value is defined to be 0 if this is not so at any vertex. An easy iterative calculation using (3.4) shows that

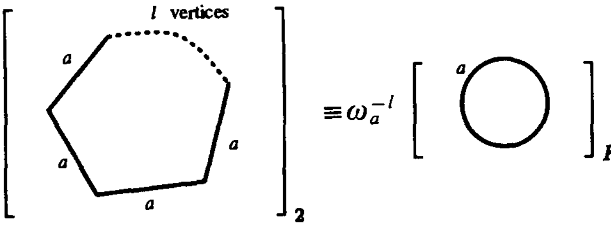


$$= (-1)^n [n+1] . \tag{3.8}$$

3.1.1. Identifying the initial data

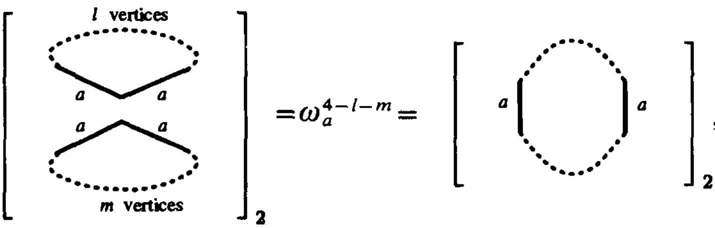
We now have all that we need in order to produce initial data for TQFTs in dimension $d=2$, 3 and 4. In each case for a positive integer r we take $q = \exp(\pi i/r)$. The colour set is then taken to be the non-negative integers less than $r-1$, $I \equiv \{0, 1, 2, ..., r-2\}$, the involution map is the identity map so that $a = a^*$, and the gluing coefficients are $\omega_a^2 \equiv (-1)^a [a+1]$. The vector spaces $V^{a_1 \cdots a_d}$ and the d-graph values, which give the maps Z, are now described for $d=2$, 3, and 4.

A $d=2$ state-sum model. Here the vector spaces V^{ab} are zero unless $a=6$ in which case they are each isomorphic to $\mathbb{C}$. Hence the value of a 2-graph is 0 unless all its edges have the same colour. We also have no need for a labelling as V^{aa} is one-dimensional. With the identification

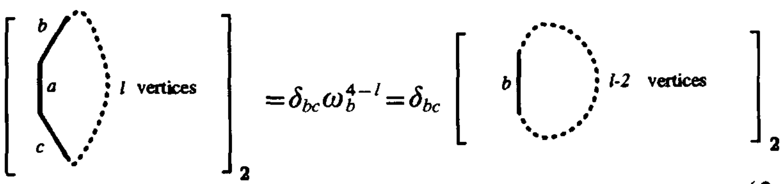


$$\equiv \omega_a^{-l} \quad = \omega_a^{2-l}, \tag{3.9}$$

the conditions (2.18) and (2.19) on 2-graph values are satisfied trivially as



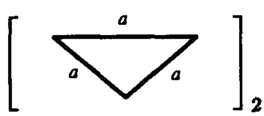
$$= \omega_a^{4-l-m} = \tag{3.10}$$



$$\sum_a \omega_a^2 \quad = \delta_{bc}\omega_b^{4-l} = \delta_{bc} \tag{3.11}$$

The maps Z are defined via (2.7) together with the identification $\hat{Z}(\Delta^2) = Z(\mathscr{D}_{\mathscr{S}^1})$, which tells us that

$$Z(\Delta^2)\cdot[aa]\otimes[aa]\otimes[aa] \equiv \quad = \omega_a^{-1}. \tag{3.12}$$



It is then easy to show via (2.7) that

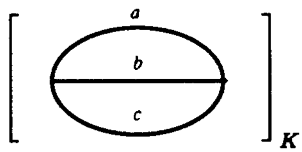
$$Z(S_{N;n_1\cdots n_p})\cdot[a] = \omega_a^{4(1-N)-2p-n_1\cdots-n_p}, \tag{3.13}$$

where $S_{N;n_1\cdots n_p}$ is a simplicial 2-manifold of genus N with p boundary components of n_1 to n_p 1-simplexes each. Note that for non-zero Z the boundary components must all be of the same colour. For closed surfaces the colour would be summed over so that for a surface of genus N one gets

$$Z(S_N) = \sum_{a\in I} \omega_a^{4(1-N)}. \tag{3.14}$$

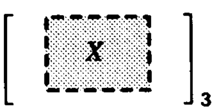
Note that the classical $r \to \infty$ limit of this agrees with the $e \to 0$ limit of the SU(2) lattice gauge theory result obtained by Witten in Ref. [17], where e was the inverse temperature. (See also Refs. [18] and [19] for more about the connection with lattice gauge theory.)

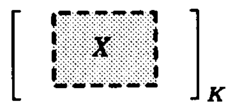
A d=3 state-sum model (Turaev–Viro theory). We now show how the Turaev–Viro model may be built using the construction of Section 2. First we denote by $\Theta(a, b, c)$ the following network value:

$\Theta(a,b,c) \equiv$ 

$$= \frac{(-1)^{(a+b+c)/2}}{[a]![b]![c]!} [\tfrac{1}{2}(a+b+c)+1]![\tfrac{1}{2}(b+c-a)]! \times [\tfrac{1}{2}(a+b-c)]![\tfrac{1}{2}(c+a-b)]! , \qquad (3.15)$$

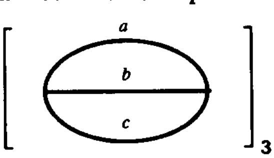
where the second identity is an explicit evaluation of the network (see appendix C of Ref. [19]). Note that by $[n]!$ we mean $[n][n-1]\cdots[2]$. Following Ref. [13] we call an admissible triple (a, b, c) r-admissible if $\Theta(a, b, c)$ is non-zero, i.e. $\frac{1}{2}(a+b+c)+1<r$. We can now define the initial data V^{abc}. If $(a+b+c)$ is r-admissible then each of V^{abc} is isomorphic to $\mathbb{C}$ and if $(a+b+c)$ is not r-admissible then V^{abc} is zero. We define 3-graph values by the following identification for general graph X:



$\equiv \prod_i (\Theta(a_i, b_i, c_i))^{-1/2}$ 

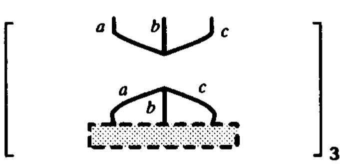
, (3.16)

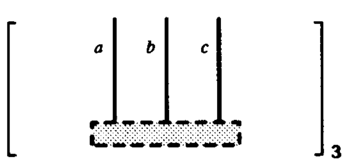
where the product is over all vertices i and (a_i, b_i, c_i) are the colours at vertex i. In particular this implies the normalisation



$=1$, (3.17)

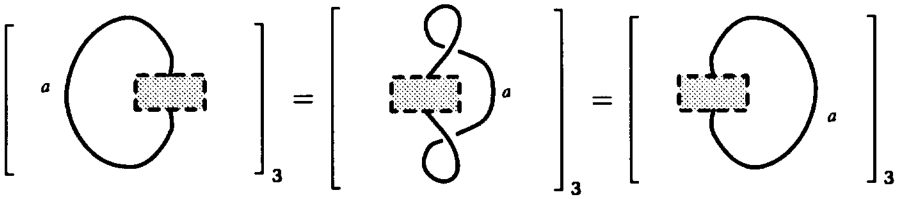
so that we have the required 3-graph identity



$=$ 

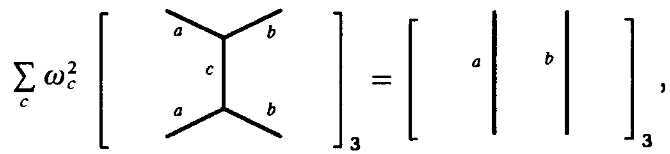
, (3.18)

which is simply saying that the only way to connect up the $(a+b+c)$ strands is as in Eq. (3.7). The identity (2.22) is a result of the regular isotopy



(3.19)

The second 3-graph identity (2.24) translates as



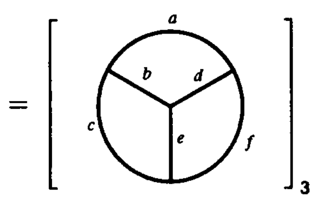
(3.20)

where the sum is over all $c \in I$ such that (a, b, c) is an r-admissible triple.

The equivalence of the TQFT defined from the above initial data and that defined by Turaev and Viro in Ref. [12] is clear when we notice the following relationship between $\hat{Z}(\Delta^3)$ and the quantum $6j$-symbols $\left|\begin{smallmatrix} i & j & k \\ l & m & n \end{smallmatrix}\right|_q$:

$$\hat{Z}(\Delta^3)\cdot[abc]\otimes[bde]\otimes[afd]\otimes[cef]$$

$$=Z(\mathscr{D}_{\mathscr{S}^2})\cdot[abc]\otimes[bde]\otimes[afd]\otimes[cef]$$



$$=D(a,b,c)D(b,d,e)D(a,f,d)D(c,e,f)$$

$$\times(-1)^{(a+b+c+d+e+f)/2}\sum_z\left(\frac{(-1)^z[z+1]!}{[t_1]![t_2]![t_3]![t_4]![t_5]![t_6]![t_7]!}\right)$$

$$=\begin{vmatrix} a/2 & b/2 & c/2 \\ e/2 & f/2 & d/2 \end{vmatrix}_q \qquad (3.21)$$

where the sum is over all z such that the t_i, which are defined as follows, are non-negative:

$$t_1 \equiv z-\tfrac{1}{2}(c+e+f)\,, \qquad t_5 \equiv \tfrac{1}{2}(a+c+d+e)-z\,,$$

$$t_2 \equiv z - \tfrac{1}{2}(a+d+f)\,, \qquad t_6 \equiv \tfrac{1}{2}(a+b+e+f) - z\,,$$

$$t_3 \equiv z - \tfrac{1}{2}(a+b+c)\,, \qquad t_7 \equiv \tfrac{1}{2}(b+c+d+f) - z\,,$$

$$t_4 \equiv z - \tfrac{1}{2}(b+d+e)\,, \tag{3.22}$$

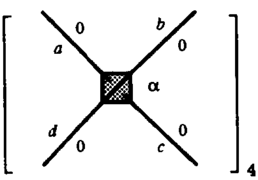
and

$$D(a,b,c) \equiv \left(\frac{[\frac{1}{2}(a+b-c)]![\frac{1}{2}(a+c-b)]![\frac{1}{2}(b+c-a)]!}{[\frac{1}{2}(a+b+c)+1]!} \right)^{1/2} . \tag{3.23}$$

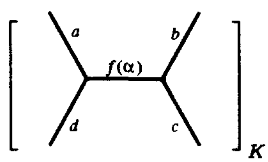
A d=4 state-sum model. In an analogous way to the $d=2$ and $d=3$ models, we define V^{abcd} to be the vector space generated by the independent ways of connecting up a Kauffman network with four legs coloured a, b, c and d. Hence V^{abcd} is isomorphic to $\mathbb{C}^{N(abcd)}$ with $N(abcd)$ equal to the number of colours $f \in I$ such that both (a, d, f) and (b, c, f) are r-admissible. It is straightforward to show that this number depends only on the 4-tuple (a, b, c, d) and not on their order.

4-graph values may be defined in terms of the Kauffman bracket by the following two relations (3.24), which is one choice of orthonormal basis, and (3.25), which defines the twist on each link.

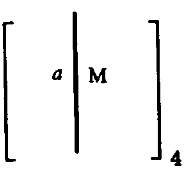
If $\max(a, d) \leqslant \min(b, c)$, define



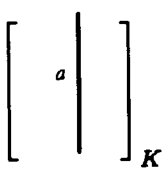
$$\equiv \frac{\omega_{f(\alpha)}\, t_{f(\alpha)}^{-1/4} (t_a t_b t_c t_d)^{1/12}}{\sqrt{\Theta(a,d,f)\Theta(b,c,f)}}$$



, (3.24)



$$\equiv (t_a)^M$$



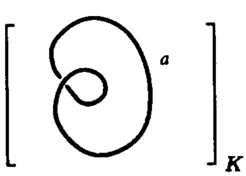
, (3.25)

where α runs from 0 to $N-1$, and

$$f(\alpha) \equiv \max(|a-d|, |b-c|) + 2\alpha\,. \tag{3.26}$$

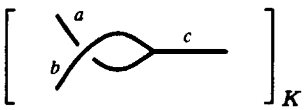
Here the twist coefficient t_a is defined by

$$t_a \equiv \omega_a^{-2}$$

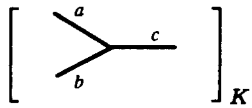


$$= (-1)^a q^{a(a+2)/2}\,. \tag{3.27}$$

More generally the following identity holds:

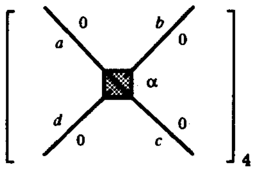


$$= (t_a t_b)^{1/2} (t_c)^{-1/2}$$

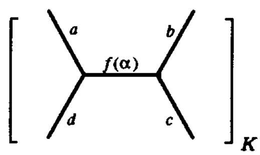


$$. \qquad (3.28)$$

We may then use Eqs. (3.24), (3.25), (3.28) together with (2.27) of Section 2 to derive the other orientations of the 4-graph vertex with $\max(a,d) \leqslant \min(b,c)$. In particular



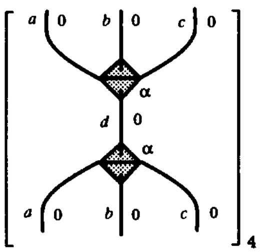
$$\equiv \frac{\omega_{f(\alpha)} t_{f(\alpha)}^{1/4} (t_a t_b t_c t_d)^{-1/12}}{\sqrt{\Theta(a,d,f)\Theta(b,c,f)}}$$



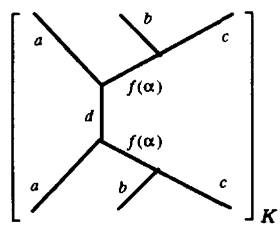
$$. \qquad (3.29)$$

Together with (3.24), this implies the 4-graph identity

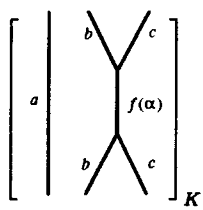
$$\sum_{\alpha,d} \omega_d^2$$



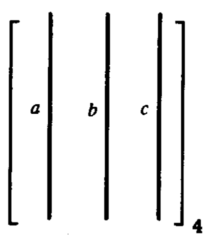
$$= \sum_{\alpha,d} \frac{\omega_d^2 \omega_{f(\alpha)}^2}{\Theta(a,d,f)\Theta(b,c,f)}$$



$$= \sum_{\alpha} \frac{\omega_{f(\alpha)}^2}{\Theta(b,c,f)}$$

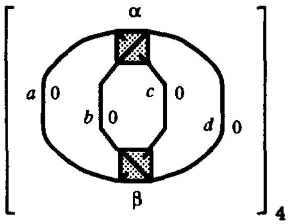


$$=$$



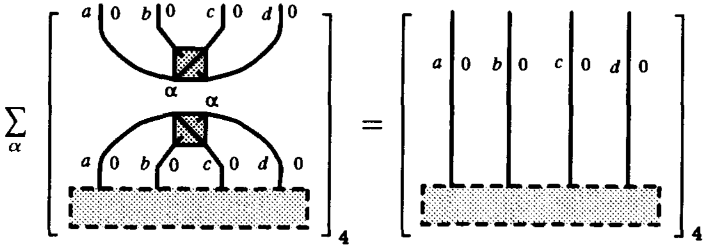
$$, \qquad (3.30)$$

where we have used relation (3.20) twice. It is also straightforward to check the orthogonality condition



$$=\delta_{\alpha,\beta}\,, \tag{3.31}$$

from which follows the second 4-graph condition



(3.32)

We see that $\hat{Z}(\Delta^4)\cdot[a]_\alpha \equiv \hat{Z}(\mathcal{D}_{\mathcal{S}^3})\cdot[a]_\alpha$ is a quantum 15*j*-symbol with $a=a_1, \ldots, a_{10}$ and $f(\alpha_1), \ldots, f(\alpha_5)$ as the 15 arguments.

4. Conclusion

We have seen that there is a very natural generalisation of the state-sum construction of Turaev and Viro to arbitrary dimension. This relies on the fact that piecewise linear topology can be understood, in all dimensions, via the equivalence relation generated by elementary moves on simplicial manifolds. Realisations of the initial data also generalise to arbitrary dimension, and we have seen that larger *q*-deformed spin-networks form the weights just as 6*j*-symbols did in the Turaev–Viro model.

Since Turaev and Viro formulated their model, clear relationships have been established with invariants of 3-manifolds constructed by other methods, in particular the Reshetikhin–Turaev invariants of Ref. [20]. It is clearly of mathematical interest to see how the $d=4$ state-sum invariants defined above are related to other 4-manifold invariants associated with classical simple Lie groups, such as the Donaldson invariants.

On the physics side it would be of great interest to see the connection between the $d=4$, $U_q sl(2)$ state-sum model and the Ashtekar variable formulation of gravity, which also has states associated with knots [21]. For more discussion on this matter as well as a description of observables in these state-sum models, the reader is referred to Ref. [19].

Acknowledgements

This work was carried out in DAMTP, Cambridge, UK, and greatly benefitted from many discussions with R.M. Williams and J. Barrett. I also gratefully acknowledge a SERC studentship.

Note added after completion

Recently Roberts [22] has noted a simple relationship between the 4-manifold invariants defined by a state-sum over $U_q sl(2)$ 15*j*-symbols in Ref. [23] (these being essentially equivalent to the invariants defined in this paper) and the signature of the 4-manifold.

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